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A Novel Variant of N-R Method and its Convergence

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Abstract

The intent of this paper is to propose a new iterative method from the well known Newton-Raphson (N-R) method for solving nonlinear equations. We also study the error estimate and the rate of convergence for the same.

Keywords: Newton-Raphson method, iterative methods, Lagrange multiplier method, order of convergence.

1. Introduction

The equation $f(x)=0$ can be rearranged in the form of a fixed point equation $g(x)=f(x)+x=x.$ The solution(s) to such type of equation is(are) computed iteratively through some iterative procedure. The Newton-Raphson ((N-R) method is one of the most popularly used iterative methods for solving such nonlinear equations. If x_{n+1} represents the (n+1)th iterate of the N-R method, then we have

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \tag{1}$$

The N-R method and other iterative methods are extensively studied by various authors and a number of variants of them appeared in the literature, see for instance, [1-3], [7-8], [10-15] and references therein. In this paper, we propose a new iterative algorithm from the N-R method and study the error estimate and rate of convergence of it.

2. New iterative Algorithm

We will use the general Lagrange multiplier method to re-derive the N-R method. If x_n is an approximate root of $f(x)=0,$ then $f(x_n)\neq 0$. We may write a correction equation in the following form

$$x_{n+1}=x_n+\lambda f(x_n), \tag{2}$$

where λ is a general Lagrange multiplier [6], which can be identified optimally by setting

$$\frac{dx_{n+1}}{dx_n}=0 \tag{3}$$

We, therefore, can identify the multiplier as follows

$$\lambda=-\frac{1}{f'(x_n)} \tag{4}$$

This on substituting into (2) gives the well-known Newton-Raphson formulation. The above idea was first proposed by Inokuti, and was further developed to the well-known variational iteration method (see [4-5] and [9]). The correction equation **(**2) may be re-written in a more general manner as follows:

$$x_{n+1}=x_n+\lambda(1+x_n)^{\alpha} f(x_n), \tag{5}$$

where α is a free parameter. Setting $\frac{dx_{n+1}}{dx_n}=0,$ we obtain

$$\lambda=-\frac{1}{\alpha(1+x_n)^{\alpha-1} f(x_n)+(1+x_n)^{\alpha} f'(x_n)}. \tag{6}$$

Hence the iteration formulation becomes

$$x_{n+1}=x_n-\frac{(1+x_n)}{\alpha f(x_n)+(1+x_n)f'(x_n)} f(x_n). \tag{7}$$

The free parameter α can be used as a control parameter to adjust convergence in every step if necessary. The optimal value for α can be determined from the relation $\frac{\partial x_{n+1}}{\partial \alpha}=0$.

3. Numerical Examples

To illustrate the effectiveness of the iterative scheme (7), we consider the following examples:

Example 1. $x^{10}-1=0,\ x_0=0.5$

Example 2. $e^{\sin x}-x=0,\ x_0=0$

Example 3 (a). $x\sin x+\cos x=0,\ \ x_0=1.0$ (b). $x\sin x+\cos x=0,\ \ x_0=0$

Example 4. $x^3 - 3x + 1 = 0,\ x_0 = 1.0$

Example 5. $x - \cos(x) = 0,\ x_0 = 4.0$

Example 6. $\sin(x) = 0,\ x_0 = 1.6.$

The following table illustrates the nature of solutions of these examples at different number of iterations.

Example	New Iterative Algorithm			Newton- Raphson Algorithm	
	α	Number of iterations	x_n	Number of iterations	x_n
1	-2	6	1.0000	42	1.0000
2	-2	3	2.2191	20	2.2191
3(a)	-2	4	2.7984	6	56.531
3(b)	-2	5	2.7984	------	Divergent
4	-2	4	1.5321	------	Divergent
5	1.5	5	0.7391	28	0.7391
6	-2	4	3.1416	7	31.4159

Table 1. Comparison between New Iterative Algorithm and N-R method

It is clear from Table 1 that the proposed scheme converges faster than N-R method and thus it is more effective in many cases.

4. Rate of convergence

Definition 4.1. If a sequence $\{x_k\}$ converges to a root R then we say it converges linearly to R if $\lim\limits_{k\to\infty} \dfrac{|x_{k+1} - R|}{|x_k - R|} = \mu$, where $0 < \mu < 1$.

Definition 4.2. If a sequence converges quadratically to a root R then $\lim\limits_{k\to\infty} \dfrac{|x_{k+1} - R|}{|x_k - R|^2} = \mu$, where $0 < \mu < 1$.

Lemma 4.1. If a root x_0 has multiplicity $k > 1$ and the orbit of the new iterative scheme (7) converges, then the orbit converges linearly.

Theorem 4.1. If a root x_0 has multiplicity 1 and the orbit of the new iterative scheme (7) converges, then the orbit converges quadratically.

Proof. To prove that the scheme (7) converges quadratically for a root of multiplicity 1, we first express $\lim_{k\to\infty}\frac{|x_{k+1}-R|}{|x_k-R|^2}$ as $\lim_{k\to\infty}\frac{|E_{k+1}|}{|E_k|^2}$, where E represents the error term. Consider the Taylor polynomial of a function $f(x)$ whose roots we wish to compute around the point x_k. Assume that $f'(x_k)\neq 0$, then

$$f(x)=f(x_k)+f'(x_k)(x-x_k)+\frac{f''(x_k)(x-x_k)^2}{2} \tag{8}$$

If $f(x)$ has a root at $x=x_r$, then from above, we easily get

$$\frac{f'(x_k)}{f(x_k)}=\frac{-f''(x_k)(x_r-x_k)}{2}-\frac{1}{x_r-x_k}$$

Now the iteration formula (7) can be written as

$$\frac{f'(x_k)}{f(x_k)}=\frac{-1}{x_{k+1}-x_k}-\frac{\alpha}{1+x_k} \tag{9}$$

and thus we get $\frac{E_{k+1}}{E_k^2-E_kE_{k+1}}=\frac{1}{2}\frac{E_kf''(x_k)}{f(x_k)}-\frac{\alpha}{1+x_k}$.

This on multiplying by $f(x_k)$ and simplifying gives

$$\frac{E_{k+1}}{E_k^2}=\frac{1}{2}f''(x_k)(x_{k+1}-x_k)-\frac{\alpha f(x_k)(x_{k+1}-x_k)}{(1+x_k)(x_r-x_k)} \tag{10}$$

If, in fact, the orbit does converge to a root then $\lim_{k\to\infty}f'(x_k)=f'(x_r)$ and $\lim_{k\to\infty}f''(x_k)=f''(x_r)$, so that $\lim_{k\to\infty}\frac{|E_{k+1}|}{|E_k|^2}=\frac{1}{2}|f''(x_k)(x_{k+1}-x_k)|>0.$

Notice that $f'(x_r)\neq 0$, because x_r has multiplicity 1. If x_r has multiplicity $k>1$ then, applying Lemma 4.1, $f(x)$ can be written as $f(x)=(x-x_r)^kG(x)$. Therefore, $f'(x)=k(x-x_r)^{k-1}G(x)+(x-x_r)^kG'(x)\Rightarrow f'(x_r)=0.$

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